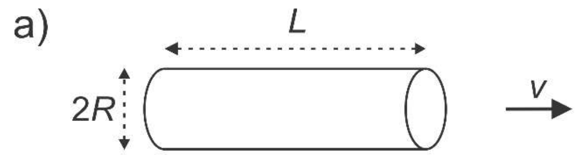
Ph.D. QUALIFYING EXAMINATION – PART A

Tuesday, January 12, 2021, 1:00 – 5:00 P.M.

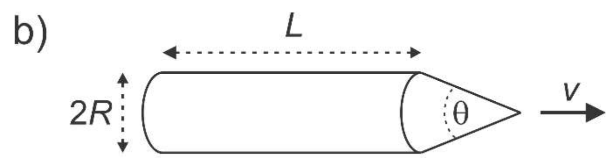
Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

A1. **Classical Mechanics: Space probe**

a) A space probe of mass M is shaped as a cylinder of radius R and length L. It moves with velocity v parallel to its symmetry axis in a cloud of dust particles of mass $m << M$ and density (particles per volume) n. The dust particles are initially at rest and collide elastically with the probe. Find the average force experienced by the probe! (The collisions are so frequent that they can be treated as continuous).



b) An improved space probe consists of a cylinder of radius R and length L and a cone-shaped tip of opening angle θ. Compute the force experienced by the improved probe in the same situation as a).



c) The speed of the probes at time $t = 0$ is v_0. Determine the subsequent time evolution of the speed.

A2. A positive charge q is fired head-on at a distant positive charge Q with an initial velocity v_0. The charge Q is held stationary. The charge q of mass m comes in, decelerates to $v = 0$, and returns out to infinity. Assume the motion is non-relativistic so that $v_0 \ll c$ and thus the power radiated is described by the Larmor formula $P = \frac{\mu_0}{6\pi c} q^2 a^2$, where a is the acceleration of the charge q. Since the energy radiated is very small, you can ignore the effect of radiative losses on the motion of the particle. Determine the fraction of the initial energy $\left(\frac{1}{2} m v_0^2\right)$ that is radiated away. You may find it easier to start from the distance of closest approach and calculate the energy radiated as the charge returns to infinity and then multiply that by two.

A3. A certain triatomic molecule forms the shape of an equilateral triangle, with an atom sitting at each vertex. An electron placed on this molecule can occupy an orbital on any one of these three atoms, in localized orthonormal quantum states which we will denote by $|1\rangle$, $|2\rangle$, and $|3\rangle$. The matrix representing the Hamiltonian H for this system has the following matrix elements:

$$\langle n|H|m\rangle = V \quad \text{for } m \neq n$$

$$\langle n|H|n\rangle = \varepsilon_0$$

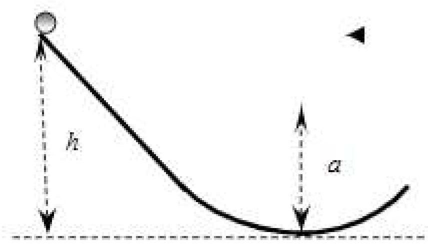
where ε_0 and V are positive constants. A measurement of the energy of the system is taken at a moment when the electron is on atom 1, i.e. when it is in the state $|\psi\rangle=|1\rangle$.

a) What possible values could be obtained in such a measurement?

b) What is the probability of finding the atom in its lowest energy state?

c) Compute the mean energy $\langle H \rangle$ and the statistical uncertainty ΔH associated with an ensemble of such measurements.

A4. A cylinder of a non-uniform radial density with mass M, length l and radius R rolls without slipping from rest down a ramp and onto a circular loop of radius a (see Fig.). The cylinder is initially at a height h above the bottom of the loop. At the bottom of the loop, the normal force on the cylinder is twice its weight.



a) Expressing the rotational inertia of the non-uniform cylinder in the general form ($I = \beta M R^2$, where β is a numerical constant), express the constant β in terms of h and a.

b) Find the numerical value of β if the radial density profile for the cylinder is given by

$\rho(r) = \rho_2 r^2$.

c) If for a cylinder of the same total mass M, the radial density profile is given by $\rho(r) = \rho_n r^n$, where $n = 0, 1, 2, \ldots$, describe qualitatively how do you expect the value of β to change with increasing n. Explain your reasoning.

A5. A star of a radius R and surface temperature T emits, approximately, as a blackbody at a rate $W = \sigma T^4$, where σ is the Stefan-Boltzmann constant. A flat solar sail of area A is used for propulsion by a spacecraft of mass M (including the sail) located at a distance r from star's center. If the sail's normal is oriented at an angle α relative to the line connecting the spacecraft and the star, what is the magnitude and the direction of its acceleration? Assume that the sail is a perfect mirror.

A6. In this problem you will be deriving the dispersion relation for an electron in a periodic lattice in a "tight-binding" type calculation. Assume that you have a string of N atoms with the spacing, b, between each atom. The string of atoms is wrapped around in a circle, so that atom N + 1 is identical to atom 1. Assume there is an extra electron added to the string of atoms and you can label the basis states of this electron as $|n\rangle$, where n is the specific atom number. If the electron is on the first atom, it is in state $|1\rangle$.

(a) Assume that you can expand the wave function of the electron over the basis states $|n\rangle$ as such: $|\psi\rangle = \sum_n c_n(t)\, |n\rangle$. Using the following notation for the matrix elements of the Hamiltonian, $H_{mn} = \langle m|H|n\rangle$, write down the time-dependent Schrodinger equation and find the set of first order differential equations that describe the time evolution of the c_n (t). You may use the orthonormality of the base states: $\langle n|m\rangle = \delta_{nm}$.

(b) If we assume that the electron can not hop from atom to atom, we expect the Hamiltonian to be diagonal, with all elements equal to E_0, and the rest zero. This will not describe a moving electron. Instead, we assume that the Hamiltonian additionally can connect nearest neighbor states, say n and n ±1, with an amplitude equal to (−A). Write down the matrix representation of the Hamiltonian for N = 6.

(c) We seek stationary state wave-like solutions to the Schrodinger equation. Take an equation for one of the $c_n(t)$'s above and, using the stationary-state time dependence with energy E, let $c_n(t) = a_n\, e^{-iEt/\hbar}$. Simplify the equation to relate the a_n's to one another.

(d) If we assume that the atom at position n has spatial coordinate $x = b_n$, and we expect wave-like solutions (electrons with wave vector k), try the solution $a_n = e^{ikx}$, and find the dispersion relation E = E(k) relating the energy to the electron wave vector. Draw a graph of this function.

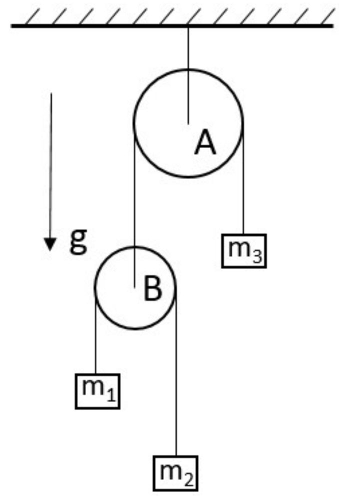
(e) What range of values can k take?

Ph.D. QUALIFYING EXAMINATION – PART B

Wednesday, January 13, 2021, 1:00 – 5:00 P.M.

Work each problem on a separate sheet(s) of paper and put your identifying number on each page. Do not use your name. Each problem has equal weight. A table of integrals can be used. Some physical constants and mathematical definitions will be provided if needed.

B1. A compound Atwood machine is composed of three blocks of masses m_1, m_2 and m_3, attached to two massless ropes of constant lengths l_A and l_B through two pulleys (both mass M) A and B of radii r_A and r_B, respectively (see figure). Pulley A is attached to a stationary ceiling, both pulleys are uniform disks. The ropes do not slip as the pulleys rotate. The system is in a constant gravitational field with acceleration g. Determine the number of the degrees of freedom, write down the Lagrangian of the system and the equations of motion. You do not need to solve them!



B2. A circular loop of radius R lies in the xy - plane with its center at the origin. The loop carries a line charge $\lambda > 0$.

a) Find the electric potential a distance z above the center of the circular loop.

b) The general solution to Laplace's equation in spherical coordinates, (r, θ, ϕ), is given by

$V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}})P_\ell(\cos\theta)$ if the problem has azimuthal symmetry (no ϕ dependence). The $P_\ell(\cos\theta)$ are the Legendre polynomials. The first three are $P_0(x) = 1$, $P_1(x) = x,\ P_2(x) = \frac{3x^2-1}{2},\ \ x = \cos\theta$.

Use your result from part (a) to determine the electric potential of the circular loop at points off the z- axis. You only need to find the first two non-zero terms for the electric potential, $V(r, \theta)$, in the upper half plane ($z > 0$).

c) A point charge Q is located at the point $(r, \theta, \phi) = (2R, \frac{\pi}{4}, \frac{\pi}{2})$. Determine the approximate force on Q using your two-term result for the potential in part (b).

B3. **Statistical Mechanics: Probability density of energy in ideal gas**

a) A cubic box of linear size L contains N particles of mass m. The particles can be considered classical non-interacting point masses. The system is in equilibrium at temperature T. Find the (properly normalized) probability density $\rho(E)$ of the energy E of a single particle! (Note that $\rho(E)\, dE$ is the probability that the single particle energy is between E and E+dE).

b) Determine the most probable single-particle energy E.

B4. A particle is in a state associated with a wave-function of the form $\psi(\vec{r}) = e^{i\vec{q}\cdot\vec{r}}\, \phi(r)$, where $\phi(r)$ is a spherically symmetric square-normalized wave function, and $\vec{q}$ is a fixed real wave-vector. Let $\langle T \rangle_\phi$ be the mean kinetic energy associated with the state $\phi(r)$.

a) Find the mean position, mean momentum, and mean kinetic energy for a particle in the state $\psi(\vec{r})$.

b) Express as a suitable integral over appropriate limits, the probability $P(r > a)$ that a position measurement will find the particle located outside a fixed radius a.

B5. Photons from a helium-neon laser with $\lambda = 632.82nm$ collide head on with incident electrons (rest mass $E_0 = 0.511MeV$) of energy $E = 10MeV$. Some of the photons are scattered directly in the backwards direction. What is the wavelength of the back-scattered photons?

B6. Fourier's law, $\vec{q} = -k\nabla T$, is exactly analogous to Ohm's Law, $\vec{J} = \sigma\vec{E}$ and $\vec{E} = -\nabla V$. In the case of Fourier's law, $\vec{q}$ is heat flux (just like $\vec{J}$ is charge flux), k is the thermal conductivity, and T is the temperature. In fact, Ohm was inspired by Fourier's work on heat conduction. Just like conservation of electric charge leads to the equation of continuity, $\nabla\cdot\vec{J} = -\partial\rho/\partial t$, conservation of energy for a passively cooling body leads to an exactly analogous equation,

$$\nabla\cdot\vec{q} = -\rho c\frac{\partial T}{\partial t},$$

where c is the specific heat of the material and ρ is its mass density.

a) Show that if the body is a sphere of uniform density and specific heat, the value of q at the surface is given by

$$q = -\frac{cM}{4\pi R^2}\frac{d\langle T\rangle}{dt},$$

where $\langle T\rangle$ is the average temperature. Hint: [Use Gauss's law to calculate the value of q on the surface in terms of its radius R and mass M. This is equivalent to finding the electric field E for uniform charge density in terms of its radius and total charge]

b) If the mean surface heat flux of the Earth (90 mW/m^2) were attributed entirely to cooling, what would be the mean rate of cooling in *K/Gyr*? Assume a specific heat of 10^3 *J/kg/K*. The radius and mass of the Earth are $6.4\cdot10^6$ *m* and $5.97\cdot10^{24}$ *kg*, respectively. Finally, there are approximately $\pi\cdot10^7$ seconds in a year.

[The total heat loss of the Earth is 47±2 TW, but the amount due simply to the passive cooling of matter in the Earth can only be estimated to be 1–17 TW. Geoneutrino experiments such as Borexino will help to constrain the contribution of radioactivity to the heat loss and, hence, constrain the cooling loss you have estimated]